Composition of Belyĭ Pairs and their Monodromy Groups

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1 Summary

A Belyĭ map $\beta : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ is a rational function with at most three critical values; we may assume these values are $\{0,\,1,\,\infty\}$. A Dessin d'Enfant is a planar bipartite graph obtained by considering the preimage of a path between two of these critical values, usually taken to be the line segment from 0 to 1. Such graphs can be drawn on the sphere by composing with stereographic projection: $\beta^{-1}\big([0,1]\big) \subseteq \mathbb{P}^1(\mathbb{C}) \simeq S^2(\mathbb{R})$. Replacing $\mathbb{P}^1$ with an elliptic curve E, there is a similar definition of a Belyĭ map $\beta : E(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$. Since $E(\mathbb{C}) \simeq \mathbb{T}^2(\mathbb{R})$ is a torus, we call (E, β) a toroidal Belyĭ pair. The corresponding Dessin d'Enfant can be drawn on the torus by composing with an elliptic logarithm: $\beta^{-1}\big([0,1]\big) \subseteq E(\mathbb{C}) \simeq \mathbb{T}^2(\mathbb{R})$.

In this project, we are interested in the group $\mathrm{Mon}(\beta) = \mathrm{im}\,\big[\pi_1\big(\mathbb{P}^1(\mathbb{C}) - \{0,\,1,\,\infty\}\big) \to S_N\big]$ called the monodromy group; it is the "Galois closure" of the group of automorphisms of the graph. With X being either $\mathbb{P}^1(\mathbb{C}) \simeq S^2(\mathbb{R})$ or $E(\mathbb{C}) \simeq \mathbb{T}^2(\mathbb{R})$, say that we have two the composition of Belyĭ maps

$$\Phi = \beta \circ \phi : \qquad X \xrightarrow{\phi} \mathbb{P}^1(\mathbb{C}) \xrightarrow{\beta} \mathbb{P}^1(\mathbb{C}) \tag{1}$$

such that $\beta\big(\{0,\,1,\,\infty\}\big) \subseteq \{0,\,1,\,\infty\}$; then the composition Φ is also a Belyĭ map. If $\mathrm{Mon}(\beta) \leq S_N$ and $\mathrm{Mon}(\phi) \leq S_M$ are the monodromy groups of β and ϕ, respectively, then $\mathrm{Mon}(\Phi) \leq S_M \wr S_N$ is a subgroup of the wreath product $S_M \wr S_N := {S_M}^N \rtimes S_N$ of the symmetric groups. We will discuss some of the challenges of determining the structure of these various groups.

2 Background

Let X be a compact, connected Riemann surface. There are two examples of interest.

- The projective line $\mathbb{P}^1$ may be embedded into the projective plane using the map $\mathbb{P}^1 \to \mathbb{P}^2$ which sends $(x_1 : x_0) \mapsto (x_1 : 0 : x_0)$, so that this curve corresponds to the zeroes of the polynomial $f(x, y) = y$. The set of complex points, namely $X = \mathbb{P}^1(\mathbb{C}) \simeq S^2(\mathbb{R})$, is a sphere.

- An elliptic curve E is a nonsingular projective variety corresponding to the zeroes of the form

$$f(x, y) = \big(y^2 + a_1\,x\,y + a_3\,y\big) - \big(x^3 + a_2\,x^2 + a_4\,x + a_6\big) = 0. \tag{2}$$

  The set of complex points, namely $X = E(\mathbb{C}) \simeq \mathbb{T}^2(\mathbb{R})$, is a torus.

A Belyĭ map $\beta : X \to \mathbb{P}^1(\mathbb{C})$ is a non-constant meromorphic function which is unbranched outside of $\{0,\,1,\,\infty\} \subseteq \mathbb{P}^1(\mathbb{C})$. Since X may be viewed as the set of zeroes of a single polynomial $f(x,y)$, we can write $\beta(x,y) = p(x,y)/q(x,y)$ as the ratio of two polynomials $p(x,y)$ and $q(x,y)$.

3 Monodromy Groups

Fix $y_0 \in \mathbb{P}^1(\mathbb{C})$ different from 0, 1, and ∞. Form the collection of affine points

$$\beta^{-1}(y_0) = \left\{ (x:y:1) \in \mathbb{P}^2(\mathbb{C}) \;\middle|\; \begin{array}{r} f(x,y) = 0 \\ p(x,y) - y_0\, q(x,y) = 0 \end{array} \right\} = \bigl\{P_1,\, P_2,\, \ldots,\, P_N\bigr\} \tag{3}$$

there exist unique paths $\widetilde{\gamma}_0^{(i)},\, \widetilde{\gamma}_1^{(i)} : [0,1] \to X$ satisfying

$$\left. \begin{array}{r} \beta\bigl(\widetilde{\gamma}_\epsilon^{(i)}(t)\bigr) = \epsilon + (y_0 - \epsilon)\, e^{2\pi\sqrt{-1}t} \\ \widetilde{\gamma}_\epsilon^{(i)}(0) = P_i \end{array} \right\} \qquad \text{where} \qquad \left\{ \begin{array}{l} P_i \in \beta^{-1}(y_0) \\ \epsilon = 0,\, 1 \end{array} \right. \tag{4}$$

There exist permutations $\sigma_0,\, \sigma_1,\, \sigma_\infty \in S_N$ such that $\widetilde{\gamma}_0^{(i)}(1) = P_{\sigma_0(i)}$, $\widetilde{\gamma}_1^{(i)}(1) = P_{\sigma_1(i)}$, and $\sigma_\infty = {\sigma_1}^{-1} \circ {\sigma_0}^{-1}$ for $i = 1,\, 2,\, \ldots,\, N$. Then $\mathrm{Mon}(\beta) = \langle \sigma_0,\, \sigma_1,\, \sigma_\infty \rangle$ is called the monodromy group of β. It is a transitive subgroup of S_N.

4 Krasner-Kaloujnine Embedding Theorem

Let $\phi : X \to \mathbb{P}^1(\mathbb{C})$ and $\beta : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ be two Belyĭ maps of degrees $M = \deg(\phi)$ and $N = \deg(\beta)$, respectively. If $\beta\bigl(\{0,\,1,\,\infty\}\bigr) \subseteq \{0,\,1,\,\infty\}$, then the composition $\Phi = \beta \circ \phi$ is a Belyĭ map of degree $M\,N$. We explain how the monodromy groups $\mathrm{Mon}(\Phi)$, $\mathrm{Mon}(\phi)$, and $\mathrm{Mon}(\beta)$ are related.

- For each $P_i \in \beta^{-1}(y_0)$, say that $\widetilde{\gamma}_\epsilon^{(ij)} : [0,1] \to X$ are those unique paths such that

$$\left. \begin{array}{r} (\beta \circ \phi)\bigl(\widetilde{\gamma}_\epsilon^{(ij)}(t)\bigr) = \epsilon + (y_0 - \epsilon)\, e^{2\pi\sqrt{-1}t} \\ \widetilde{\gamma}_\epsilon^{(ij)}(0) = P_{ij} \end{array} \right\} \qquad \text{where} \qquad \left\{ \begin{array}{l} P_{ij} \in \phi^{-1}(P_i) \\ \epsilon = 0,\, 1 \end{array} \right. \tag{5}$$

  Then $\widetilde{\gamma}_\epsilon^{(i)} = \phi \circ \widetilde{\gamma}_\epsilon^{(ij)}$ are those unique paths $\widetilde{\gamma}_0^{(i)},\, \widetilde{\gamma}_1^{(i)} : [0,1] \to \mathbb{P}^1(\mathbb{C})$ satisfying

$$\left. \begin{array}{r} \beta\bigl(\widetilde{\gamma}_\epsilon^{(i)}(t)\bigr) = \epsilon + (y_0 - \epsilon)\, e^{2\pi\sqrt{-1}t} \\ \widetilde{\gamma}_\epsilon^{(i)}(0) = P_i \end{array} \right\} \qquad \text{where} \qquad \left\{ \begin{array}{l} P_i \in \beta^{-1}(y_0) \\ \epsilon = 0,\, 1 \end{array} \right. \tag{6}$$

  Observe that $\widetilde{\gamma}_\epsilon^{(ij)}(1) = P_{IJ}$ where $I = \sigma_\epsilon(i)$ and $J = \tau_\epsilon^{(i)}(j)$ for some $\sigma_\epsilon \in S_N$ and $\tau_\epsilon^{(i)} \in S_M$. Hence we have the following well-defined elements of the wreath product $S_M \wr S_N = {S_M}^N \rtimes S_N$:

$$\bigl(\tau_\epsilon^{(1)},\; \tau_\epsilon^{(2)},\; \ldots,\; \tau_\epsilon^{(N)},\; \sigma_\epsilon\bigr) \qquad \text{for } \epsilon = 0,\, 1. \tag{7}$$

- We have a surjective projection map from $G = \mathrm{Mon}(\beta \circ \phi)$ to $\mathrm{Mon}(\beta)$ whose kernel $H = \ker\big[\mathrm{Mon}(\beta \circ \phi) \twoheadrightarrow \mathrm{Mon}(\beta)\big]$ contains $\mathrm{Mon}(\phi)$ embedded diagonally.

$$\begin{tikzcd} 1 \arrow[r] & {S_M}^N \arrow[r] & S_M \wr S_N \arrow[r] & S_N \arrow[r] & 1 \\ & \mathrm{Mon}(\phi) \arrow[r] \arrow[u, hook] & \mathrm{Mon}(\Phi) \arrow[r] \arrow[u, hook] & \mathrm{Mon}(\beta) \arrow[u, hook] & \end{tikzcd} \tag{8}$$

  In particular, G must be a subgroup of $H \wr (G/H)$. (This may be viewed as a special case of the Krasner-Kaloujnine Embedding Theorem.)

5 Examples on the Sphere

Say that $X = \mathbb{P}^1(\mathbb{C}) \simeq S^2(\mathbb{R})$.

- The rational function $\beta(z) = 4\,z\,(1-z)$ is a Belyĭ map of degree $N = 2$ which satisfies $\beta\big(\{0,\,1,\,\infty\}\big) \subseteq \{0,\,1,\,\infty\}$. The monodromy group has the generators

$$\begin{aligned} \sigma_0 &= \big(1\ 2\big) \\ \sigma_1 &= \big(1\big) \\ \sigma_\infty &= \big(1\ 2\big) \end{aligned} \tag{9}$$

  Hence the monodromy group is $\mathrm{Mon}(\beta) = \langle \sigma_0,\, \sigma_1,\, \sigma_\infty \rangle = S_2$, the symmetric group of degree 2.

- The rational function $\phi(z) = -(z-1)\,(2\,z^2 + 3\,z + 9)^3/729$ is a Belyĭ map of degree $N = 7$. According to our `Mathematica` code, the monodromy group has the generators

$$\begin{aligned} \sigma_0 &= \big(1\ 5\ 3\big)\,\big(2\ 4\ 6\big) \\ \sigma_1 &= \big(3\ 7\ 4\big) \\ \sigma_\infty &= \big(1\ 3\ 2\ 6\ 4\ 7\ 5\big) \end{aligned} \tag{10}$$

  Hence the monodromy group is $\mathrm{Mon}(\phi) = \langle \sigma_0,\, \sigma_1,\, \sigma_\infty \rangle = A_7$, the alternating group of degree 7.

- The composition $\Phi = \beta \circ \phi$ is the rational function

$$\Phi(z) = -\frac{4}{531441}\,(z-1)\,z^3\,\big(2\,z^2 + 3\,z + 9\big)^3\,\big(8\,z^4 + 28\,z^3 + 126\,z^2 + 189\,z + 378\big) \tag{11}$$

  which is a Belyĭ map of degree $N = 14$. According to our `Mathematica` code, the monodromy group has the generators

$$\begin{aligned} \sigma_0 &= \big(3\ 7\ 5\big)\ \big(4\ 6\ 8\big)\ \big(11\ 13\ 12\big) \\ \sigma_1 &= \big(1\ 3\big)\ \big(2\ 4\big)\ \big(5\ 11\big)\ \big(6\ 12\big)\ \big(7\ 9\big)\ \big(8\ 10\big)\ \big(13\ 14\big) \\ \sigma_\infty &= \big(1\ 5\ 12\ 4\ 2\ 8\ 10\ 6\ 13\ 14\ 11\ 7\ 9\ 3\big) \end{aligned} \tag{12}$$

Hence the monodromy group is $\mathrm{Mon}(\Phi) = \langle \sigma_0,\, \sigma_1,\, \sigma_\infty \rangle = \big(A_7 \times A_7\big) \rtimes Z_2$, the wreath product of A_7 by S_2.

6 Examples on the Torus

Say that $X = E(\mathbb{C}) \simeq \mathbb{T}^2(\mathbb{R})$.

- For any positive integer n, the square of the nth Chebyshev polynomial

$$\beta(x) = T_n(x)^2 = \cos^2\big(n \cdot \arccos(x)\big) = \begin{cases} x^2 & \text{for } n = 1 \\ \big(2\,x^2 - 1\big)^2 & \text{for } n = 2 \\ x^2\,\big(4\,x^2 - 3\big)^2 & \text{for } n = 3 \end{cases} \tag{13}$$

  is a Belyĭ map of degree $N = 2\,n$ which satisfies $\beta\big(\{0,\,1,\,\infty\}\big) \subseteq \{0,\,1,\,\infty\}$. When $n = 1$, $1 - \beta(1 - 2\,z) = 4\,z\,(1 - z)$, so that $\mathrm{Mon}(\beta) = Z_2$, the cyclic group of order 2.

- Consider the elliptic curve

$$E:\ y^2 = x\,(x-1)\,(x-\lambda) \qquad \text{where} \qquad \lambda = \cos\frac{\pi}{2\,n}. \tag{14}$$

  Then $\Phi(x, y) = \beta(x)$ is a Belyĭ map of degree $N = 4\,n$. Say that $n = 1$. According to our `Mathematica` code, the monodromy group has the generators

$$\begin{aligned} \sigma_0 &= (1\ 3)\ (2\ 4) \\ \sigma_1 &= (1\ 2\ 3\ 4) \\ \sigma_\infty &= (1\ 2\ 3\ 4) \end{aligned} \tag{15}$$

  Hence the monodromy group is $\mathrm{Mon}(\Phi) = \langle \sigma_0,\, \sigma_1,\, \sigma_\infty \rangle = Z_4$, the cyclic group of order 4.

7 Future Work

We would like to know more about the structure of $G = \mathrm{Mon}(\beta \circ \phi)$. In particular, we would like to know more about how $H = \ker\big[\mathrm{Mon}(\beta \circ \phi) \twoheadrightarrow \mathrm{Mon}(\beta)\big]$ is related to $\mathrm{Mon}(\phi)$.

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